Quantum Physics 1
Class 18

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Commutation Relations

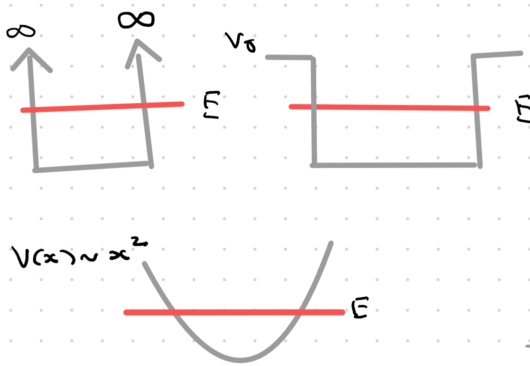
Last Time:

1.) $\hat{A}\varphi_a = a\varphi_a$

eigenvalue.

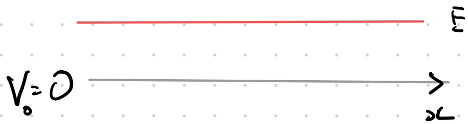
φ_a can be a "discrete" function,

for example



discrete

can also be continuous:



continuous

Consider $\frac{\hbar}{i}\frac{\partial}{\partial x}$ → momentum operator with eigenfunctions (independent of $V(x)$) e^{ikx} [ie the basis]

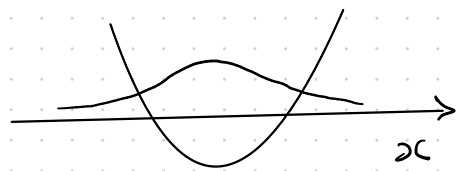
$$\psi(x) = \sum_n c_n \varphi_n(x)$$

discrete

$$\psi(x) = \int A(p)\, e^{i\frac{p}{\hbar}x} \qquad \underline{\underline{\text{continuous}}}$$

where $|c_n|^2$, $|A(p)|^2 \sim$ probability.

example: Q.H.O.

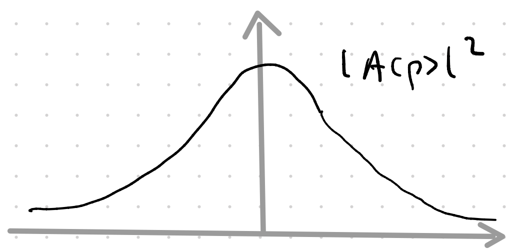


$$\varphi_0(x) = N e^{-\frac{m\omega}{2\hbar}x^2}$$
(ground state)

Now note that: $\varphi_0(x) = \int A(p)\, e^{-\frac{ipx}{\hbar}}\, dp$

where $A(p) = \int e^{-\frac{m\omega}{2\hbar}x^2} e^{-ip/\hbar\, x}\, dx$

$\underline{NB}$: $A(p) \sim$ a distribution of p.



<u>Hermitian Operators</u>

$$\langle A^\dagger \rangle = \int \psi^* A^\dagger \varphi(x)\, dx = \int (A\psi(x))^* \varphi(x)\, dx$$

$$= \int \psi^*(x) A \varphi(x)\, dx \quad \text{if } A^\dagger = A$$

that is, the eigenvalues are real.

Commutation Relations

Defn : $\boxed{[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}}$

example, $[\hat{x}, \hat{p}]$:

evaluate above w/t :

$$\left[x \frac{\hbar}{i}\frac{\partial}{\partial x} - \frac{\hbar}{i}\frac{\partial}{\partial x} x\right]\varphi(x)$$

$$= \frac{\hbar}{i} x \frac{\partial}{\partial x}\varphi(x) - \frac{\hbar}{i}\left[x \frac{\partial \varphi(x)}{\partial x} + \varphi(x)\frac{\partial x}{\partial x}\right]$$

$$= -\hbar/i \; \varphi(x)$$

$$= i\hbar\, \varphi(x)$$

$$\therefore \left[x \frac{\hbar}{i}\frac{\partial}{\partial x} - \frac{\hbar}{i}\frac{\partial}{\partial x} x\right] = i\hbar$$

$$\boxed{[\hat{x}, \hat{p}] = i\hbar}$$

$\therefore$ $\hat{x}$ & $\hat{p}$ do not commute example : $[\hat{x}, \hat{x}]$

$$\Rightarrow \quad [\hat{x}, \hat{x}]\,\varphi(x) = x^2 \varphi(x) - x^2 \varphi(x)$$
$$= 0 \qquad [\text{ this commutes }]$$

example : $[\hat{p}, \hat{p}]$

$$\Rightarrow \quad [\hat{p}, \hat{p}]\,\varphi(x) = \left[\frac{\hbar}{i}\frac{\partial}{\partial x}, \frac{\hbar}{i}\frac{\partial}{\partial x}\right]\varphi(x)$$
$$= 0 \qquad [\text{this commutes}]$$

In-class 18-1

(#1) Commuting Hermitian operators possess common eigenfunctions (but different eigenvalues)

$$\boxed{[\hat{A}, \hat{B}] = 0} \quad \Rightarrow \quad \hat{A}\hat{B} - \hat{B}\hat{A} = 0$$
$$\therefore \quad \hat{A}\hat{B} = \hat{B}\hat{A} \quad \ldots \text{(i)}$$

Now, $\hat{A}\,\varphi_a(x) = a\,\varphi_a(x)$

$$\hat{B}\,\hat{A}\,\varphi_a(x) = a\,\hat{B}\,\varphi_a(x)$$

use ① : $\underbrace{\hat{A}\hat{B}\varphi_a(x)} = a\underbrace{\hat{B}\varphi_a(x)}$

$\therefore$ we can say, $\hat{B}\varphi_a(x)$ also eigenvector of $\hat{A}$.

Note: $\hat{B}\varphi_a(x) = b\,\varphi_a(x)$; $\varphi_a(x)$ is eigenfunc. of $\hat{B}$ w/t eigenvalue "b".

(#2) Non-commuting Operators:

$$[\hat{A}, \hat{B}] \neq 0$$

$$\Rightarrow [\Delta\hat{A}, \Delta\hat{B}] \geq \frac{|\langle\hat{C}\rangle|}{2}$$

$$\Rightarrow \Delta x\,\Delta p \geq \hbar/2$$

(#3)
$$\boxed{\frac{d\langle A\rangle}{dt} = \frac{i}{\hbar}\langle[\hat{H}, \hat{A}]\rangle}$$

Now, $\hat{H}\Psi(x,t) = i\hbar\frac{\partial}{\partial t}\Psi(x,t)$

$\&$ $[\hat{H}\Psi(x,t)]^* = -i\hbar\frac{\partial}{\partial t}\Psi^*(x,t)$

Now, $\frac{d}{dt}\langle A\rangle = \frac{d}{dt}\int \Psi^* \hat{A}\, \Psi \, dx$

$$= \int \frac{d}{dt}\left(\Psi^* \hat{A}\, \Psi \right) dx$$

$$\Rightarrow \frac{d}{dt}\left(\Psi^* \hat{A} \Psi\right) = \Bigg[\frac{\partial \Psi^*}{\partial t}\left(\hat{A}\Psi\right) + \underbrace{\Psi^* \frac{\partial}{\partial t}\left(\hat{A}\Psi\right)}_{\Psi^* \hat{A} \frac{\partial \Psi}{\partial t} + \Psi \cancelto{0}{\frac{\partial A}{\partial t}}} \Bigg]$$

$$\Rightarrow \frac{d}{dt}\left(\Psi^* \hat{A} \Psi\right) = \left[\frac{i}{\hbar}\left(\hat{H}\Psi\right)^* \hat{A}\, \Psi + \Psi^* \hat{A} \frac{-i}{\hbar} \hat{H}\, \Psi \right]$$

$$= \frac{i}{\hbar}\left[\left(\hat{H}\Psi\right)^* \hat{A}\Psi - \Psi^* \hat{A} \hat{H}\, \Psi \right]$$

In-class 18.2, 18.3, 18.4.