Quantum Physi cs 1
Class 15

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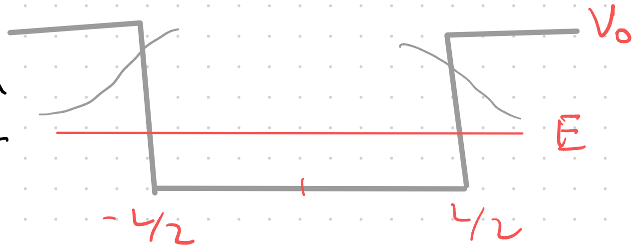
Scattering from a step potential

Last Time:

Finite Square Well:

$x > \frac{L}{2}$; $e^{-\kappa_1 x}$

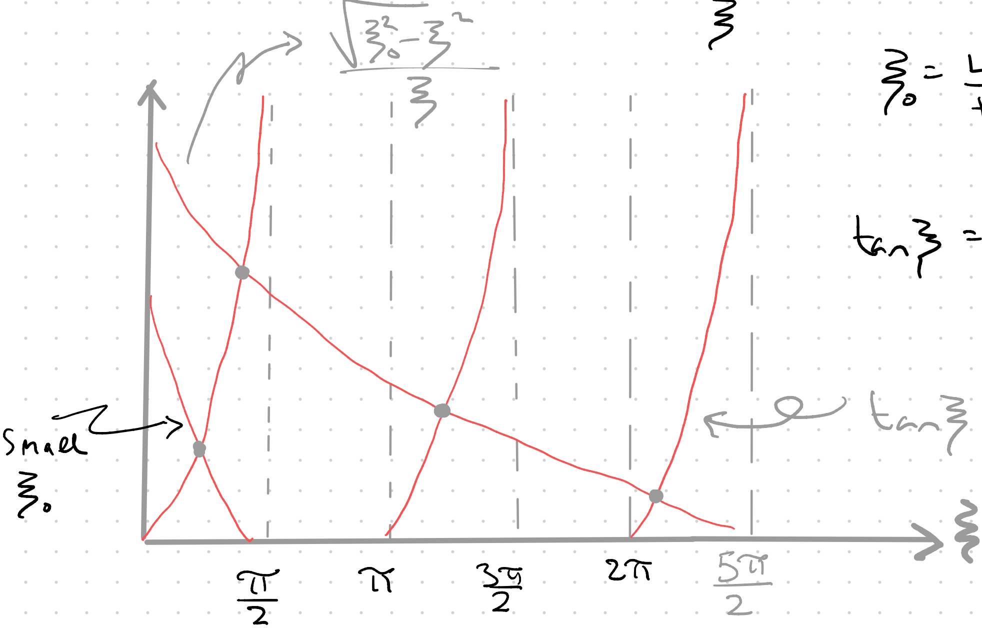
$$\kappa_1 = \frac{\sqrt{2m(V_0-E)}}{\hbar}$$



$x < -\frac{L}{2}$; $e^{+\kappa_1 x}$

E solutions from: $\tan\xi$, $\frac{\sqrt{\xi_0^2-\xi^2}}{\xi}$ $\qquad \xi = \frac{\sqrt{2mE}}{\hbar}$

$$\xi_0 = \frac{L}{\hbar}\sqrt{mV_0/2}$$

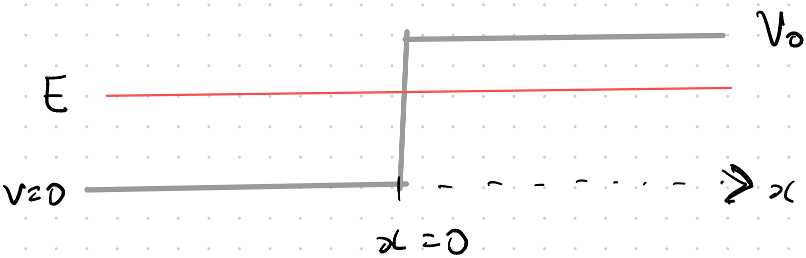


$$\tan\xi = \frac{\sqrt{\xi_0^2-\xi^2}}{\xi}$$

- Even functions $\quad n=1 \quad$ ground state: $\cos kx$

What happens when $E > V_0$?

Consider the following:

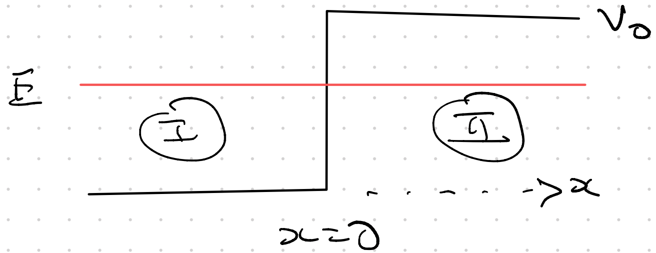


time independ. Schrödinger Equation:

$$-\frac{\hbar^2}{2m}\frac{d^2\varphi(x)}{dx^2} + V(x)\varphi(x) = E\varphi(x)$$

$$\Rightarrow \frac{d^2\varphi(x)}{dx^2} = -\frac{2m}{\hbar^2}(E-V)\varphi(x) = -k_0^2\,\varphi(x)$$

where $k_0 = \frac{\sqrt{2m(E-V_0)}}{\hbar}$, $k = \frac{\sqrt{2mE}}{\hbar}$



Now consider region I and region II:

Region I: for $x<0$; $E > (V=0)$

$$k = \frac{\sqrt{2mE}}{\hbar} > 0 \quad ; \quad \text{solutions: } Ae^{ikx} + Be^{-ikx}$$

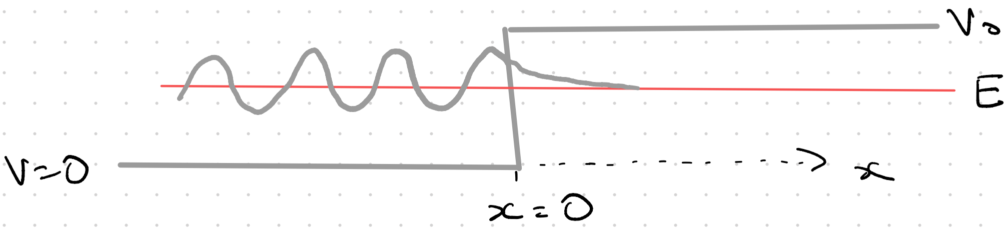
Region II: for $x>0$; $E < V_0$.

$$k_0 = \frac{\sqrt{2m(E-V_0)}}{\hbar} = \frac{i\sqrt{2m(V_0-E)}}{\hbar}$$

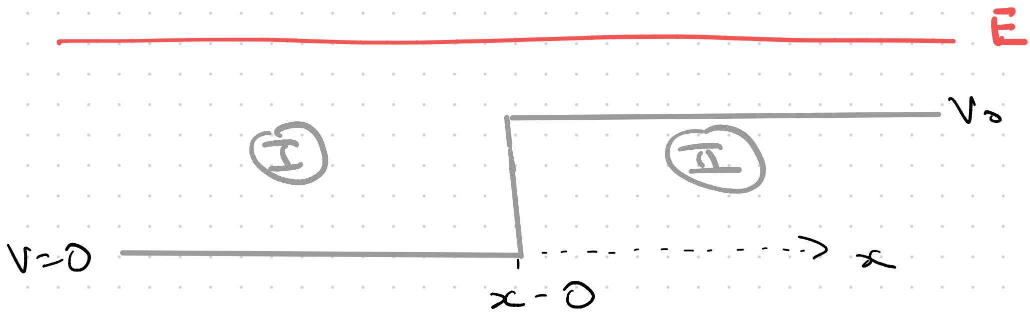
$$\& \quad K_1 \equiv \frac{\sqrt{2m(V_0-E)}}{\hbar}$$

$$\therefore \varphi = Ce^{ik_0x} + De^{-ik_0x}$$

$$\varphi = Ce^{-K_1x} + \underbrace{\cancel{D}}_{0}e^{+K_1x}$$



③ What happens when $E > V$ everywhere?



Region I: $x<0$; $k = \frac{\sqrt{2mE}}{\hbar} > 0$

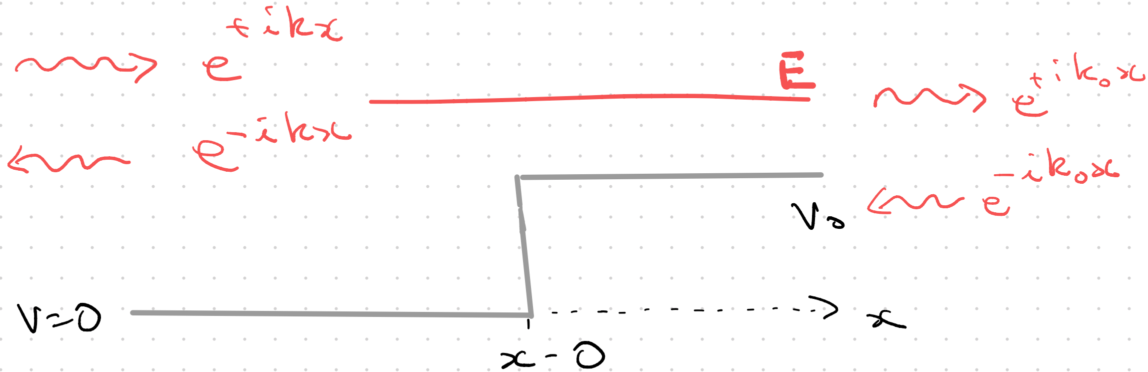
w/ solution $\varphi \sim Ae^{ikx} + Be^{-ikx}$

Region II: $x>0$; $E > V_0$, $k_0 = \frac{\sqrt{2m(E-V_0)}}{\hbar}$

with solutions:

$$\varphi \sim Ce^{ik_0x} + De^{-ik_0x}$$

Consider incident travelling wave from the left:



e^{+ikx} ⇒ flux coming from the left

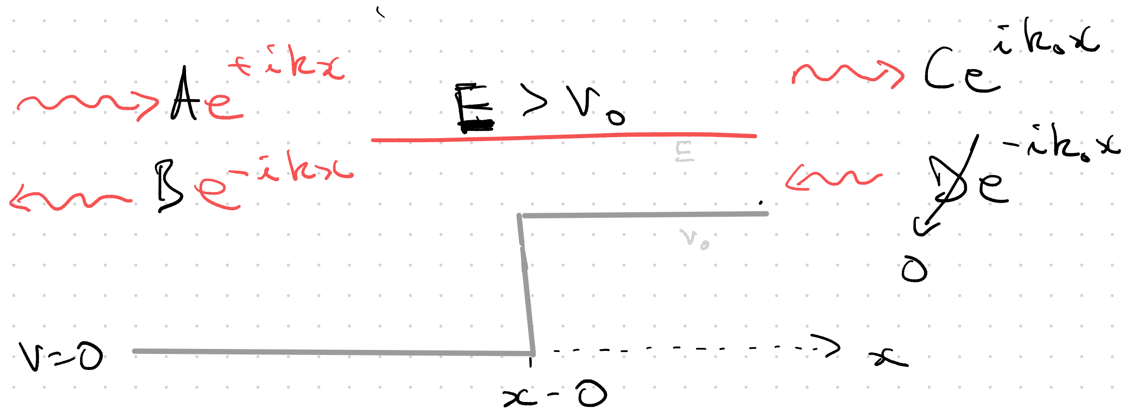
e^{-ikx} ⇒ flux reflected from the potential

→ Region I

$$\therefore \begin{cases} x<0 & \varphi(x) = A e^{ikx} + B e^{-ikx} \; ; \; k = \sqrt{\dfrac{2mE}{\hbar^2}} \\ x>0 & \varphi(x) = C e^{ik_0 x} + \cancel{D} e^{-ik_0 x} \; ; \; k_0 = \dfrac{\sqrt{2m(E-V_0)}}{\hbar} \end{cases}$$

$D \to 0$

[If no wave incident from $x = +\infty$]

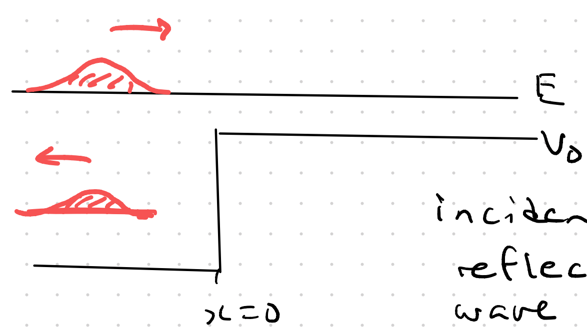


There is reflection and transmission:

Let's use the probability current to quantify this:

recall: $$j_x = \frac{\hbar}{2mi}\left(\varphi^* \frac{\partial \varphi}{\partial x} - \varphi \frac{\partial \varphi^*}{\partial x}\right)$$

Now, $R = \dfrac{j_{ref}}{j_{inc}}$; $T = \dfrac{j_{trans}}{j_{inc}}$



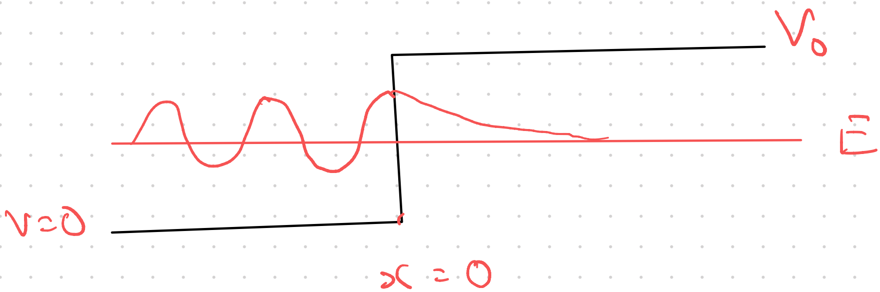
incident and reflected wave packets,

recall: $\psi_{(x,t)} = \int A(k)\, e^{i(kx-\omega t)}\, dx$

In-class: 15.1, 15.2, 15.3

Now let's consider scattering for the case

$E \ngtr V_0$ everywhere:



$$-\frac{\hbar^2}{2m}\frac{d^2\varphi(x)}{dx^2} + V(x)\,\varphi(x) = E\,\varphi(x)$$

(1) $x < 0$; $k = \frac{\sqrt{2mE}}{\hbar}$; $E > (V=0)$

solutions: $\varphi \sim A e^{+ikx} + B e^{-ikx}$

Now $x > 0$, $k_0 = \frac{\sqrt{2m(E-V_0)}}{\hbar} = i\frac{\sqrt{2m(V_0-E)}}{\hbar}$

w/t solutions: $\varphi \sim Ce^{ik_0 x} + De^{-ik_0 x}$

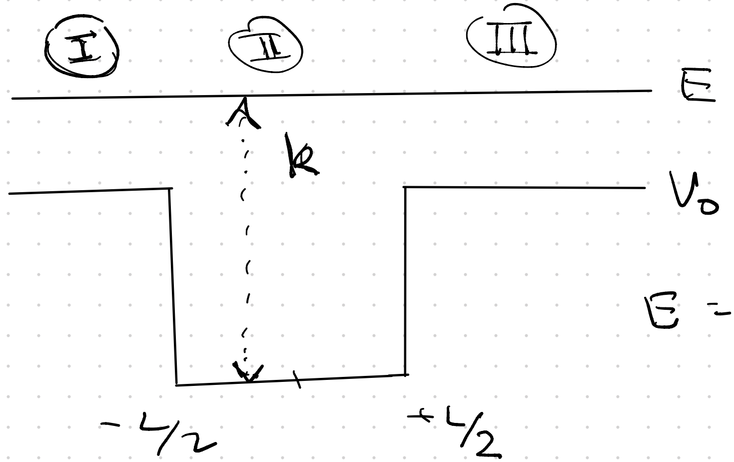
$$\varphi \sim Ce^{-k_1 x} + \cancel{D}e^{+k_1 x}$$

$\rightarrow 0$ to avoid unphysical solutions.

What is the transmission and the reflection probability?

$\boxed{\text{In-class 14.4}}$

Reconsider case where $E > V$ everywhere



$E = K + V(x)$

Region I: $E = \frac{\hbar^2 k_0^2}{2m} + V_0$; $k_0 = \frac{\sqrt{2m(E-V_0)}}{\hbar}$

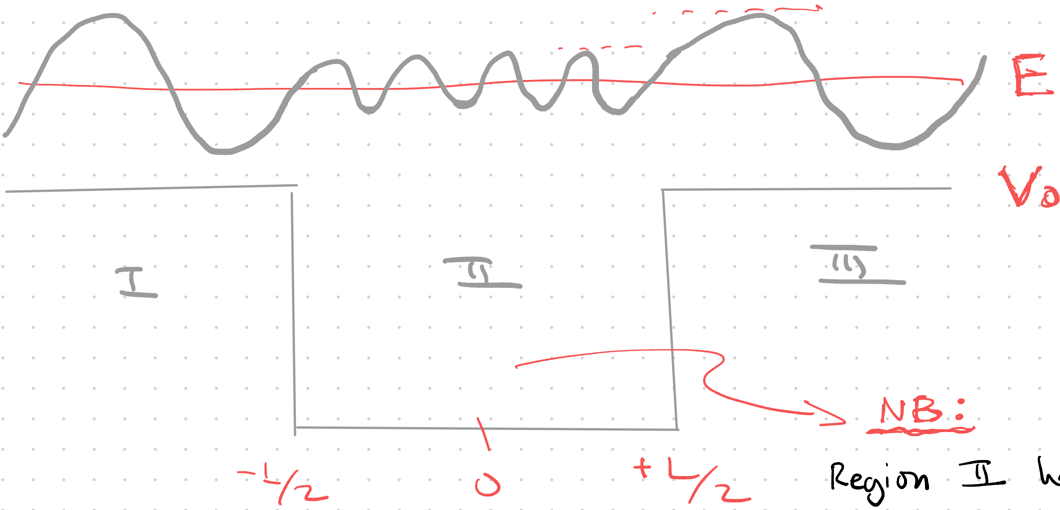
k_0 is small, $\lambda \to$ large.

Region III: $E = \frac{\hbar^2 k_0^2}{2m} + V_0$ [Same as region I]

Region II: $E = \frac{\hbar^2 k^2}{2m}$; $k = \sqrt{\frac{2mE}{\hbar^2}}$

k is large, λ is small

Solutions: $e^{\pm ikx}$, $e^{\pm ik_0 x}$



NB:

Region II has smaller amplitude and smaller $|\psi|^2$

Aside:

consider: $\psi \sim A \sin(kx)$ ... (1)

$$k = \sqrt{\frac{2m(E-V)}{\hbar^2}}$$

From (1)

$$\psi = A \sin(k_0 x)$$

$$\Rightarrow |\psi|^2 = A^2 \sin^2(k_0 x) \quad \ldots (3)$$

$$\frac{d\psi}{dx} \sim kA\cos(kx) \quad \ldots (2)$$

$$\frac{1}{k^2}\left(\frac{d\psi}{dx}\right)^2 \sim k^2 A^2 \cos^2(kx)\frac{1}{k^2}$$

$$\frac{1}{k^2}\left(\frac{d\psi}{dx}\right)^2 \sim A^2\cos^2(kx) \quad \ldots (4)$$

Add (3) + (4) $\Rightarrow |\psi|^2 + \frac{1}{k^2}\left(\frac{d\psi}{dx}\right)^2$

$$\sim A^2\left(\sin^2(kx) + \cos^2(kx)\right)$$

$$\therefore \quad A = \left[|\psi|^2 + \frac{1}{k^2}\left(\frac{d\psi}{dx}\right)^2 \right]^{1/2}$$

when $A \downarrow$; $k \uparrow$ & $\lambda \downarrow$