Quantum Physics 1

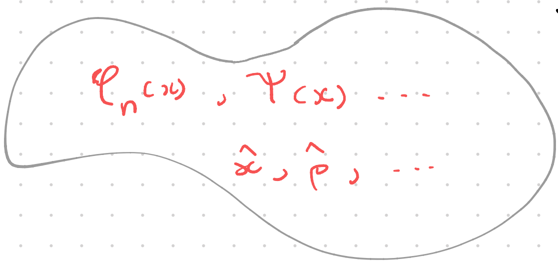
Class 12

Class 12

Energy Eigenvalue Problems

Last Time :

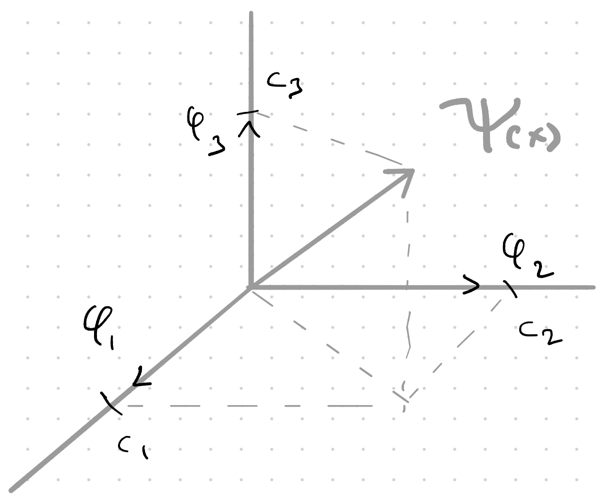
Functional Vector Space



* When the vector space is square-integrable, it is known as a "Hilbert space"

$$\int_{-\infty}^{\infty} \Psi^*_{(x)} \Psi_{(x)} \, dx = 1$$

** Generalization of 2D, 3D Eucliedean space, which describes a vector space of finite or infinite # of dimensions.



$$\Psi_{(x)} = \sum_n c_n \varphi_n(x)$$

VIP Properties :

① Normalizable

$$\int \varphi^*_n(x) \varphi_n(x) \, dx = 1 ; m = n$$

② Orthogonal

$$\int \varphi^*_m(x) \varphi_n(x) \, dx = 0 , m \neq n$$

③ Can project $\Psi(x)$ onto $\varphi_m(x)$ basis

$$\int \varphi_m^* \Psi(x)\, dx = \int \varphi_m^*(x) \sum_n c_n \varphi_n(x)\, dx$$
$$= c_m$$

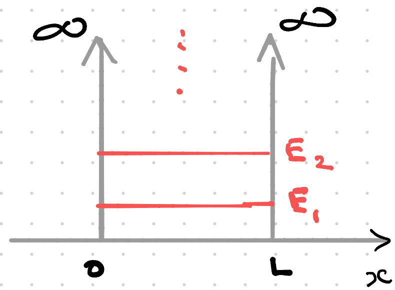
Consider the inner product :

Hilbert Space	Euclidean Space
$\Psi(x) = \sum c_n \varphi_n(x)$	$\vec{A} = \sum_n A_n \hat{\eta}_n = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$
$\Phi(x) = \sum D_n \varphi_n(x)$	$\vec{B} = \sum_n B_n \hat{\eta}_n = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$
$\int \Psi^*(x) \Phi(x)\, dx$	$\vec{A} \cdot \vec{B} = (\sum A_n \hat{\eta}_n) \cdot (\sum B_n \hat{\eta}_n)$
$= \int \sum_m \sum_n c_n^* D_m \delta_{mn}$	$= (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \cdot (B_x \hat{i} + B_y \hat{j} + B_z \hat{k})$
$= \sum_m c_m^* D_m$	$= A_x B_x + A_y B_y + A_z B_z$

Consider the infinite square well :

$$\varphi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n \pi x}{L}\right) \equiv \text{Basis vector}$$

(solution to time indep. S.E.)



where S.E. :

$$\underbrace{\left[-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + V(x) \right]}_{\hat{H} \text{ or energy operator}} \varphi(x) = E \varphi(x)$$

Aside: classically,

$$\hat{H} = p^2/2m + V(x)$$

Revisit the problem from a slightly different perspective :

$$\hat{H}\,\varphi(x) = \overbrace{E\,\varphi(x)}^{\text{NB: A constant} \times \text{eigenf.}}$$

energy operator

eigenfunction

constant number, eigenvalue.

Generally :

eigen function.

$$\hat{A}\,\varphi_a = a\,\varphi_a$$

operator

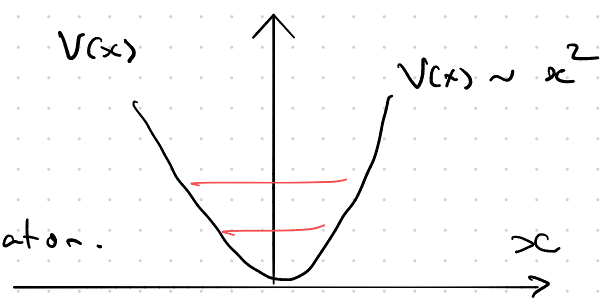
eigenvalue.

where $\hat{A}$ can be $\hat{H}$, $\hat{p}$, $\hat{x}$, ...

Examples of other basis other than that of the ∞ square well :

① Harmonic oscillator. Where difference is the form of $V(x)$

$\varphi_n \equiv$ basis for the Harmonic oscillator.



<u>NB</u> For the eigenfunction / eigenvalue relationship to hold, operator acting on the eigenfunction should yield an eigenfunction times a number.

example:

$$\hat{p}\left(\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)\right) = \frac{\hbar}{i}\frac{\partial}{\partial x}\left(\sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right)\right)$$

$$= \frac{\hbar}{i}\sqrt{\frac{2}{L}}\,\frac{n\pi}{L}\cos\left(\frac{n\pi x}{L}\right)$$

$$\neq \text{constant} \times \sin\left(\frac{n\pi x}{L}\right)$$

$\therefore$ $\sin\left(\frac{n\pi x}{L}\right)$ is <u>not</u> an eigenfunction of $\hat{p}$

$$\boxed{\text{In-class } 12.1,\ 12.2}$$

NB: $\hat{H}$ is a linear operator.

consider: $\psi(x) = c_1 \varphi_1(x) + c_2 \varphi_2(x)$

Now if $\hat{H} \varphi_n(x) = E_n \varphi_n(x)$

Then
$$\hat{H} \psi_n(x) = \hat{H} \left[c_1 \varphi_1(x) + c_2 \varphi_2(x) \right]$$
$$= c_1 E_1 \varphi_1(x) + c_2 E_2 \varphi_2(x)$$
$$\neq E \left[c_1 \varphi_1(x) + c_2 \varphi_2(x) \right]$$

in general unless $E_1 = E_2$

$\boxed{\text{In-class 12-3, 12.4}}$

What are the eigenfunctions for the operator $\hat{p}$?

Recall: $\hat{p} = \frac{\hbar}{i} \frac{\partial}{\partial x}$

eigenfunction

$$\Rightarrow \underbrace{\frac{\hbar}{i} \frac{\partial}{\partial x}}_{\text{operator}} \varphi(x) = p \, \varphi(x)$$

eigenvalue.

guess: $\varphi(x) \sim e^{ikx}$

$$\Rightarrow \quad \frac{\hbar}{i}\frac{\partial}{\partial x} e^{ikx} \rightarrow \frac{\hbar}{i}(ik)\, e^{ikx}$$

$$\rightarrow \hbar k e^{ikx}$$

$$\therefore \quad \boxed{p = \hbar k} \text{ , eigenvalue of } \hat{p}.$$

$\therefore$ w/t $\varphi(x)$ as bases (eigenfunction)

$$\Psi(x) = \int A(k)\, e^{ip/\hbar\, x}\, dk$$

* integral because k is continuous.