Quantum Physics 1

Class 10

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Functional vector space

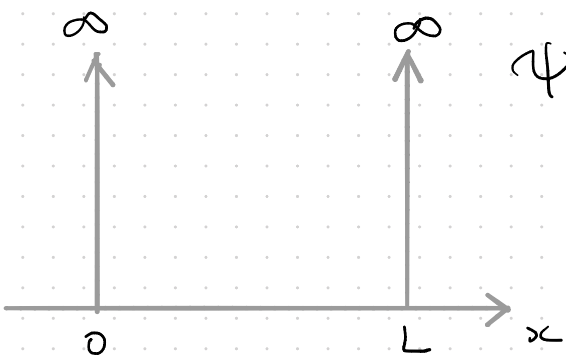
Last Time:

We solved S.E. using separation of variables

$$\Psi(x,t) = \psi(x)\, f(t)$$

$$\begin{cases} \dfrac{\hbar^2}{2m} \dfrac{\partial^2 \psi(x)}{\partial x^2} + V(x)\,\psi(x) = E\,\psi(x) \quad \ldots (1) \\[2ex] \dfrac{\partial f}{\partial t} = -i \dfrac{E}{\hbar} f(t) \ldots (2) \\[2ex] \Rightarrow f(t) = f(0)\, e^{-iEt/\hbar} \end{cases}$$

Infinite Square Well:



normaliz. const.

$$\psi(x) = \sqrt{\frac{2}{L}} \sin(k_n x)$$

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2}$$

Kinetic energy.

VIP points:

(1) Energy is quantized
(2) Ground state energy > 0
(3) $\psi^* \psi$ is time independent.

$\boxed{\text{In-class } 10.1, 10.2, 10.3}$

Recall superposition:

$$\Psi(x,t) = \sum c_n \varphi_n(x) e^{-iE_n t/\hbar}$$

at $t=0$ $\Psi(x,0) = \sum c_n \varphi_n(x)$ ** Fourier Series Expansion

↳ what is c_n?

$$\Rightarrow \Psi(x,0) = \sum c_n \varphi_n(x)$$

multiply both sides by φ_m^*

$$\Rightarrow \varphi_m^* \Psi(x,0) = \sum c_n \varphi_m^* \varphi_n(x)$$

$$\Rightarrow \int \varphi_m^* \Psi(x,0)\, dx = \sum c_n \underbrace{\int \varphi_m^* \varphi_n(x)\, dx}$$

$$\underbrace{\int \varphi_m^* \varphi_n \, dx}$$

$$\int \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \begin{cases} 0 & m \neq n \\ 1 & m = n \end{cases}$$

$$= \delta_{m,n} \quad \text{Kronecker delta.}$$

$$\Rightarrow \int \varphi_m^* \Psi(x,0)\, dx = \sum c_n \delta_{m,n}$$

$$= c_m$$

$$\therefore \quad c_m = \int \varphi_m^* \Psi(x,0)\, dx \equiv \text{Fourier coefficient.}$$

** <u>NB</u> : $|c_m|^2 \sim$ probability of a particle being in the m^{th} state.

Now, recall for $t > 0$

$$\Psi(x,t) = \sum_n c_n \varphi_n(x) e^{-iE_n t/\hbar}$$

Consider: $\int_0^L \Psi^*(x,t)\, \Psi(x,t)\, dt = 1$

$$= \int_0^L \sum_n c_n^* \varphi_n^*(x) \sum_m c_m \varphi_m(x)\, dx$$

$$= \sum_n \sum_m c_n^* c_m \delta_{m,n}$$

$$= \sum_m |\underbrace{c_m}| ^2 = 1$$

probability of finding particle in the m^{th} state

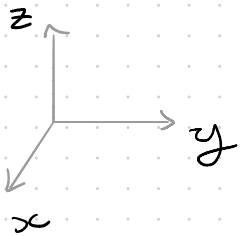
In-class 10.4

Functional Vector Space

Recall Geometrical vector space.

vectors: $\vec{A}, \vec{B}, \dots$

in Euclidean space



$$\vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$

$$= \sum A_i \hat{\eta}_i \; ; \quad \hat{\eta}_1 = \hat{i}$$

$$\hat{\eta}_2 = \hat{j}$$

$$\hat{\eta}_3 = \hat{k}$$

recall:

(i) $\hat{\eta}_i \cdot \hat{\eta}_j = \delta_{ij}$

(ii) $A_{\hat{i}} = \vec{A} \cdot \hat{\eta}_i$

$$= A_x$$

compare with:

Wavefunction vector space

with objects:

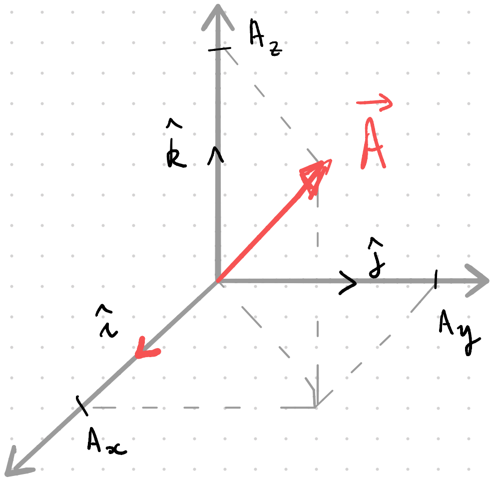
$$\Psi(x,t), \; \varphi_n, \; \varphi_m^*$$

$\varphi_n \Rightarrow$ unit vectors (basis vectors)

$$\Psi(x) = \sum c_n \varphi_n(x)$$

$$c_n = \int \varphi_n^* \Psi(x) \, dx$$

Geometrical vector space



Now,

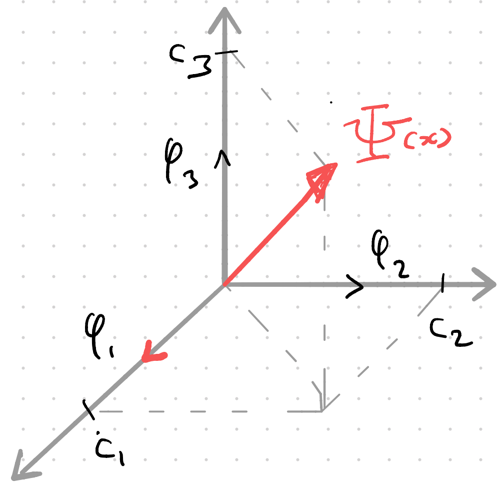
$$\hat{\eta}_i \to \varphi_n$$

$$A_i \to c_n$$

$$\begin{cases} \vec{A} \cdot \hat{\eta}_i = A_x \\ c_n = \int \varphi_n^*(x)\, \Psi(x)\, dx \end{cases}$$

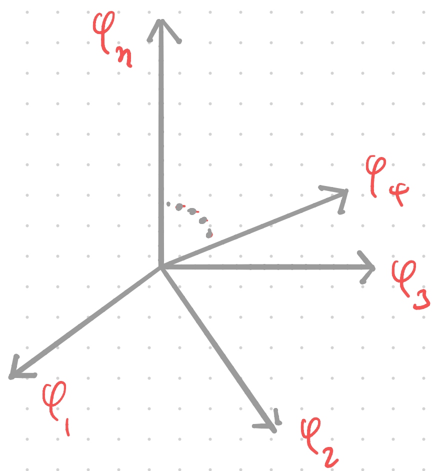
$$\begin{cases} \hat{\eta}_i \cdot \hat{\eta}_j = \delta_{ij} \\ \int \varphi_m^* \varphi_n \, dx = \delta_{m,n} \end{cases}$$

Wavefunction vector space:



in general:

$\exists$ a High-dimensional space



Inner product:

$$\vec{A} = \sum A_n \hat{\eta}_n = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}$$

$$\vec{B} = \sum_n B_n \hat{\eta}_n = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}$$

$$\& \quad \vec{A} \cdot \vec{B} = \left(\sum_n A_n \hat{\eta}_n\right) \cdot \left(\sum_m B_m \hat{\eta}_m\right)$$

$$= A_x B_x + A_y B_y + A_z B_z$$

[Dot product]

$$\Psi(x) = \sum c_n \varphi_n(x)$$

$$\Phi(x) = \sum D_n \varphi_n(x)$$

$$\& \int \vec{\Psi}^{*}(x) \, \Phi(x)$$

$$= \int \sum_m \sum_n c_n^{*} \varphi_n^{*}(x) D_n \varphi_n(x) \, dx$$

$$= \sum_m \sum_n c_m^{*} D_n \delta_{mn}$$

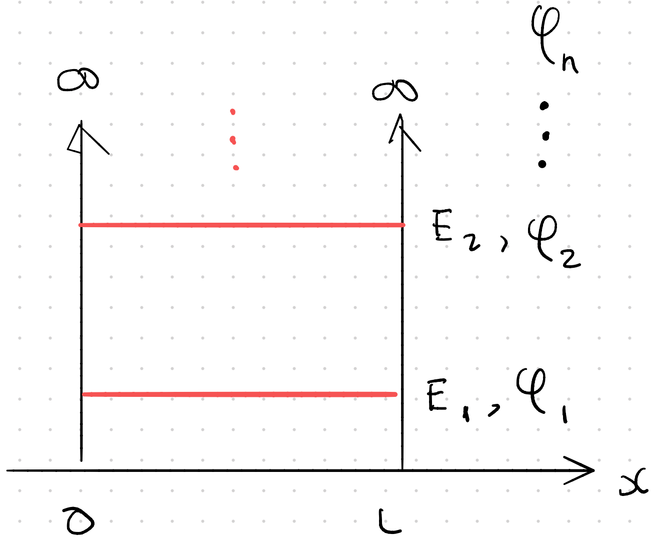
$$= \sum_n c_n^{*} D_n$$

Now, reconsider the time independent Schrödinger equation & infinite square well.

$$\Rightarrow \varphi_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n \pi x}{L}\right) \rightarrow \boxed{\text{basis vector}}$$

$$\Rightarrow \underbrace{\left[\frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x)\right]}_{\equiv \hat{H}} \varphi(x) = E \varphi(x)$$

[Aside: Classically $\hat{H} = \frac{p^2}{2m} + V$]



where $\varphi_n(x) \equiv$ Eigenfunctions of the Hamiltonian $\hat{H}$