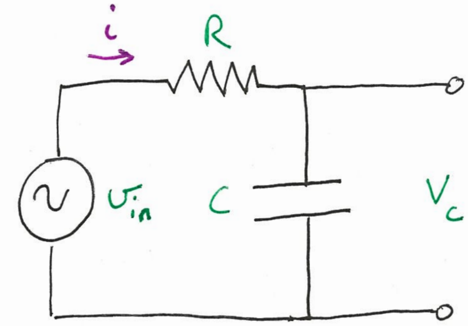
Lecture 19 - RC, RL, RLC ac circuits

The RC circuit (exponential analysis)

We want to find the voltage amplitude V_c across the capacitor.



Procedure:

- Apply voltage $v_{in} = V_0 \cos(\omega t)$
- Use complex notation $v_{in} = V_0 \,\mathrm{Re}(e^{j\omega t})$ temporarily dropping the "Re".
- Write down the voltage loop rule using $v_x = i Z_x$ for each circuit element.
- Add impedance in circuits just like resistance.
- Solve for i; solve for v_c as a function of ω.

In circuit above with $v_{in} = V_0 e^{j\omega t}$:

- Loop rule: $v_{in} - v_c - v_R = v_in - iZ_c - iZ_R = 0$

$$v_{in} = i\left(\frac{1}{j\omega C} + R\right)$$

- $$i(t) = \frac{v_{in}}{R + 1/j\omega C} = \frac{v_{in}\,(R - 1/j\omega C)}{(R + \frac{1}{j\omega C})(R - 1/j\omega C)} = v_{in}\,\frac{R + j/\omega C}{(1/\omega C)^2 + R^2}$$

Note: Re() in-phase with driving voltage. Im() is $-\frac{\pi}{2}$ out of phase.

- $I = (i \cdot i^*)^{1/2} = v_{in} \dfrac{1}{\left[R^2 + \left(\frac{1}{wC}\right)^2\right]^{1/2}}$

- $v_c = iX_c = v_{in} \dfrac{\left(R + \frac{j}{wC}\right)\left(-\frac{1}{wC}\right)}{R^2 + \left(\frac{1}{wC}\right)^2}$

Interpretation:

There are two parts to the current response, i.e., in- and out of phase with the applied voltage.

The signal current:

$$i = v_{in} \frac{R + j/wC}{R^2 + \left(\frac{1}{wC}\right)^2} = \frac{v_{in}}{R} \frac{(RwC)^2 + jwC}{1 + (RwC)^2}$$

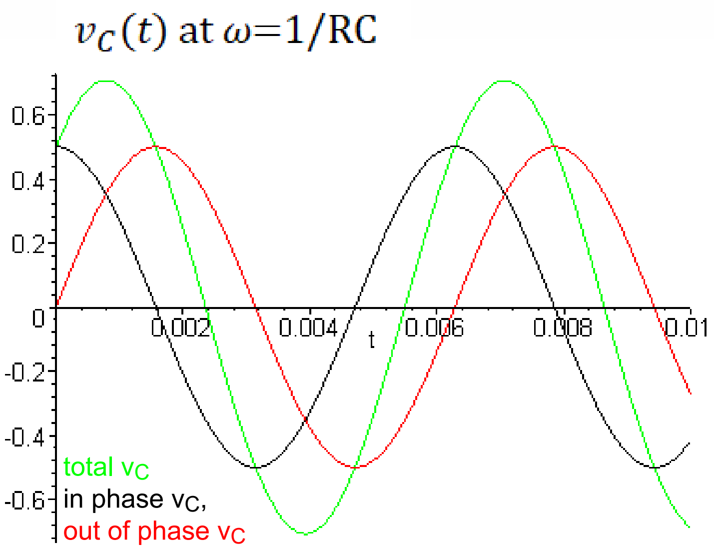
is nearly constant at high frequency and is resistor-like because the impedance of the capacitor becomes very small.

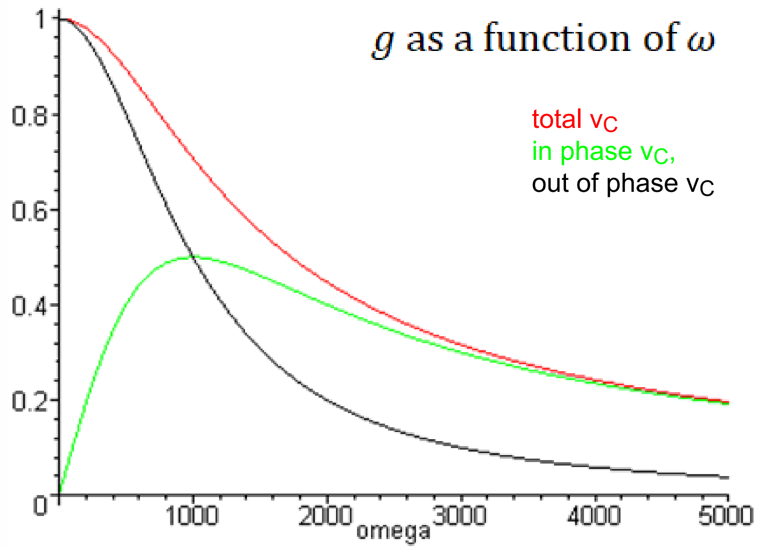
At low frequency, the current is controlled by the high impedance of the capacitor.

The overall capacitor voltage magnitude is

$$V_c \text{ GAIN} = g = \left|\frac{V_c}{V_{in}}\right| = \left|\frac{IX_c}{V_{in}}\right| = \left|\frac{1}{(1 + jwRC)}\right| = \sqrt{\frac{1}{1 + (wRC)^2}}$$

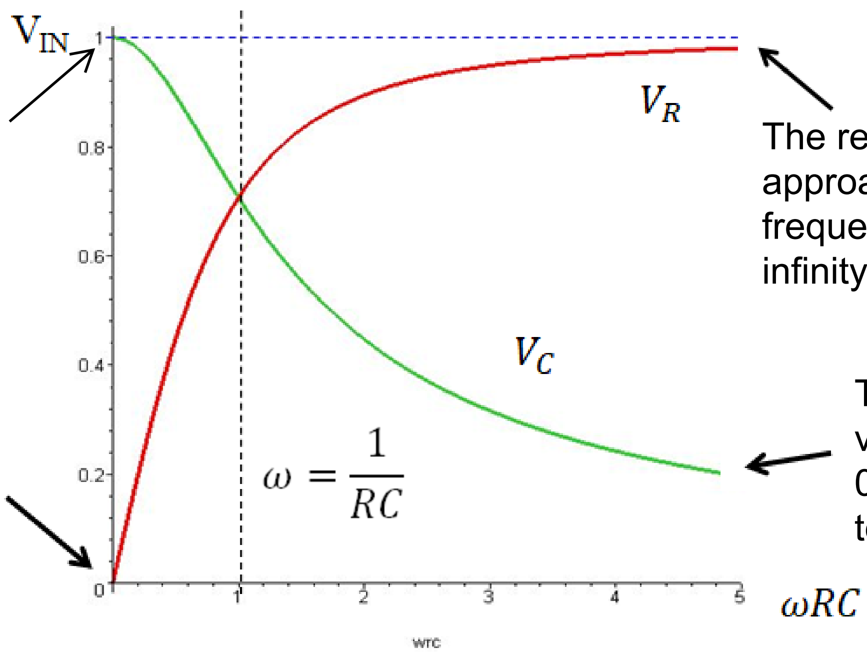
Voltage across the capacitor in an RC circuit





The capacitor voltage approaches V_{IN} as frequency goes to 0

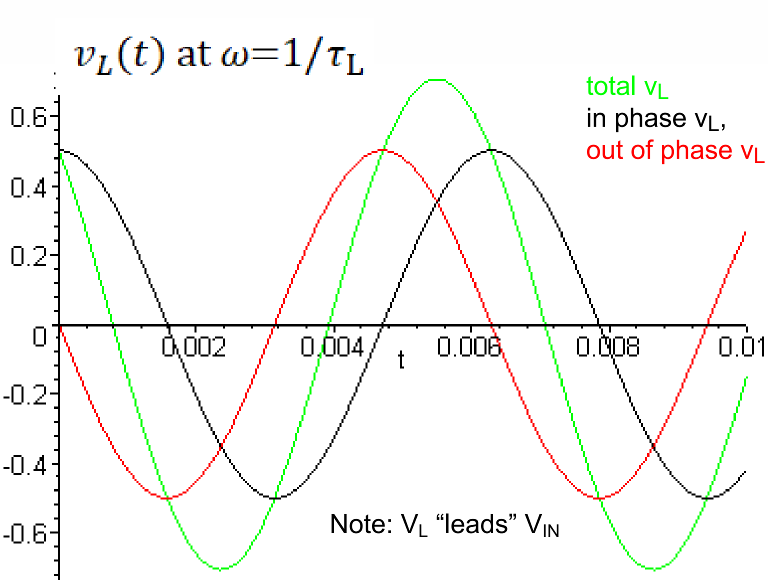
The resistor voltage approaches V_{IN} as frequency goes to infinity.

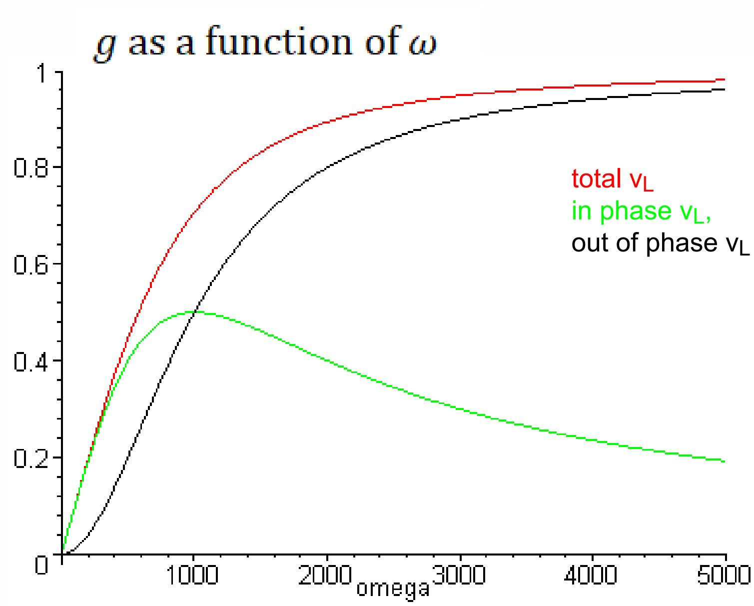


The resistor voltage approaches 0 as frequency goes to 0.

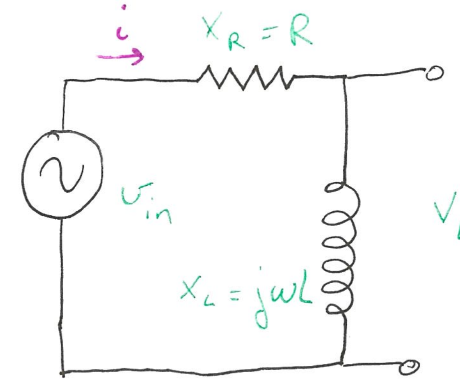
The capacitor voltage approaches 0 as frequency goes to infinity

Voltage across the capacitor in an RL circuit





The RL circuit (exponential analysis)



We want to find the voltage amplitude V_L across the inductor.

Procedure:

- Apply voltage $v_{in} = V_0 \cos(\omega t)$
- Write down voltage loop rule using $v_k = i Z_k$ for each circuit element.
- Solve for i, then I.
- Solve for V_L as a function of ω.

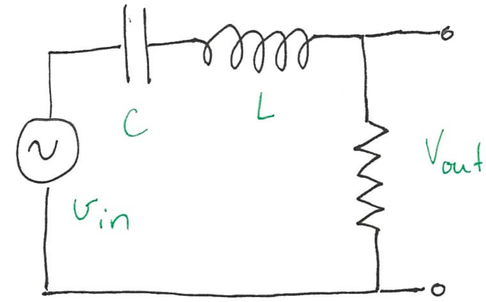
Here: $v_{in} = iZ = iR + j\omega L)$

$$\Rightarrow \quad i(t) = \frac{v_{in}}{R + j\omega L} = \frac{v_{in}/R}{(1 + j\omega L/R)}$$

$$v_L(t) = i Z_L = i j\omega L = \frac{v_{in} j\omega L/R}{1 + j\omega L/R} = \frac{v_{in}(j\omega \tau_L + \omega^2 \tau_L^2)}{(1 + \omega^2 \tau_L^2)}$$

$$g_L = \frac{V_L}{V_{in}} \frac{\omega \tau_L}{\sqrt{(1 + (\omega \tau_L)^2)}} \qquad \text{with} \quad \tau_L = \frac{L}{R}$$

The RLC circuit (exponential analysis)



Loop equation using impedance:

$$V_{in} e^{j\omega t} = \frac{1}{j\omega C} i + j\omega L i + R i$$

$$\Rightarrow \quad i(t) = \frac{V_{in} e^{j\omega t}}{-\frac{j}{\omega C} + j\omega L + R}$$

$$V_{out} = R i = V_{in} e^{j\omega t} \frac{R}{j\left(\omega L - \frac{1}{\omega C}\right) + R}$$

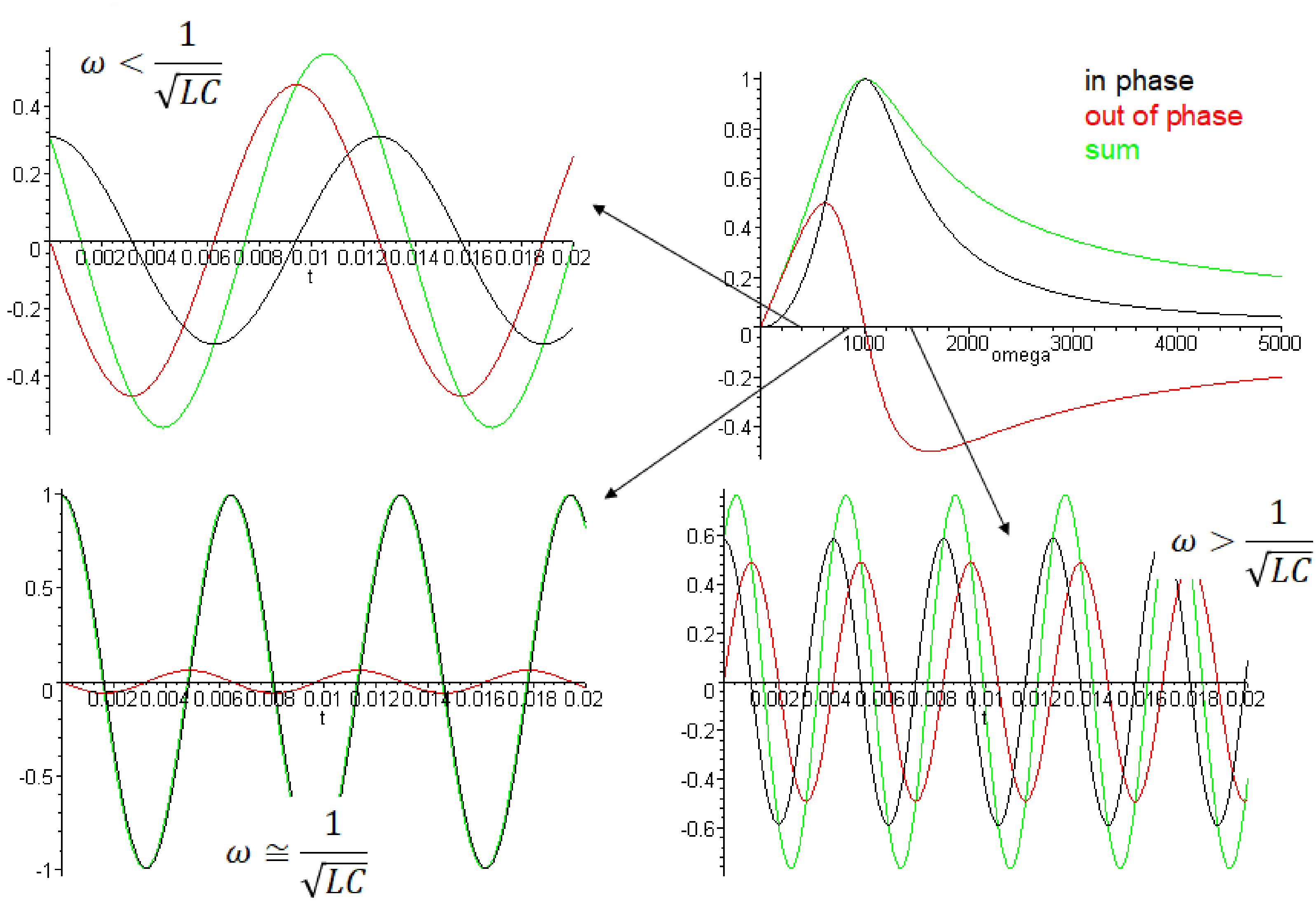
$$\frac{V_{out}}{V_{in}} = \frac{1 - j\left(\omega \frac{L}{R} - \frac{1}{\omega RC}\right)}{1 + \left(\omega \frac{L}{R} - \frac{1}{\omega RC}\right)^2} \quad ; \quad g = \sqrt{\frac{R/L}{\frac{R}{L} + \left(\omega - \frac{1}{\omega LC}\right)^2}}$$

1) When $\omega^2 = \frac{1}{LC} \Rightarrow \omega = \sqrt{\frac{1}{LC}}$ then $\frac{V_{out}}{V_{in}} = 1$

2) When $\omega \rightarrow 0$, then $\frac{V_{out}}{V_{in}} \approx -j\omega RC$

3) When $\omega \rightarrow \infty$, then $\frac{V_{out}}{V_{in}} \approx \frac{j}{\omega L/R}$

RLC current behavior



Resonances in the driven RLC circuit

At any instant : $v_d(t) = v_R + v_C + v_L = V_d \cos(\omega_d t)$

but v_R, v_C, and v_L have different phases and must be added carefully.

$$I = \frac{V_d}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{V_d}{\sqrt{R^2 + \left(\omega_d L - \frac{1}{\omega_d C}\right)^2}}$$

When the driving frequency is such that $\left(\omega_d L - \frac{1}{\omega_d C}\right) = 0$, then

$I = \frac{V_d}{R}$ and is a maximum.

The smaller R is, the sharper the maximum.