Lecture 18 - AC circuits and Phasors

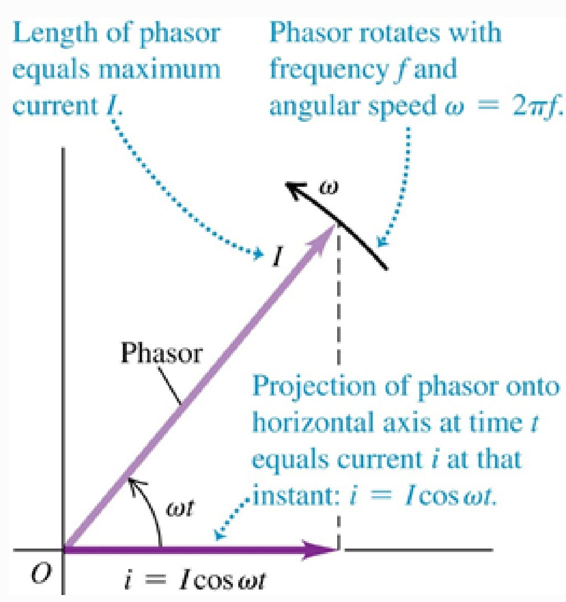
Why are harmonic (sinusoidal) signals useful?

- Harmonic emfs occur when we spin a coil in a magnetic field.
- Sinusoidal voltages and currents can be easily transformed using pairs of inductors.
- Any periodic signal can be represented as the sum of sine and cosine signals (Fourier series).
- Any pulse can be represented as the integral of sine and cosine signals (Fourier transform).

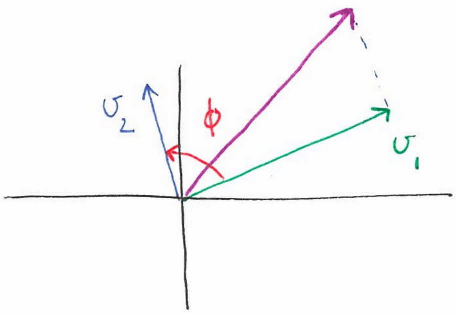
Phasor representation

A harmonic quantity $v = V_0 \cos(\omega t + \phi)$ can be represented by a rotating vector known as a phasor using the following conventions:

1. Phasors rotate in the CCW direction with angular speed ω.
2. The length of each phasor is proportional to the AC quantity amplitude.
3. The projection of the phasor on the x-axis gives the instantaneous value



When we want to add instantaneous voltages around a loop, we need to keep the phase in mind.

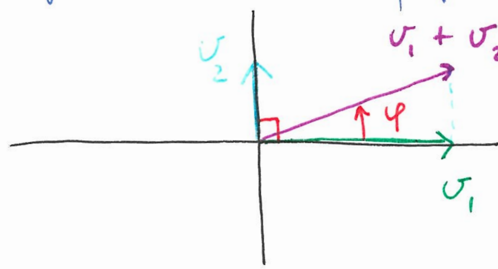


$$v_1(t) = V_1 \cos(\omega t)$$

$$v_2(t) = V_2 \cos(\omega t + \phi)$$

Add phasors just like vectors.

Example: adding two AC voltages 90° out of phase



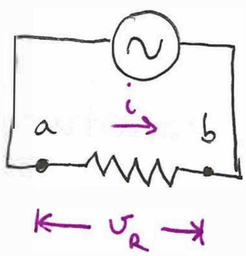
$$v_1(t) = V_1 \cos(\omega t)$$

$$v_2(t) = V_2 \cos\left(\omega t + \frac{\pi}{2}\right)$$

$$v_1(t) + v_2(t) = V_T \cos(\omega t + \varphi)$$

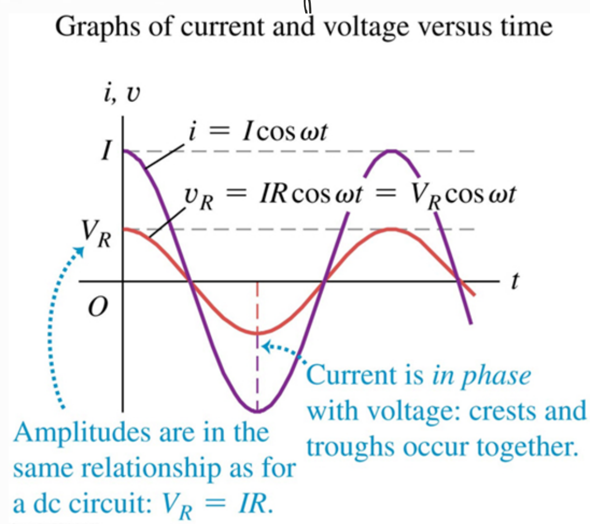
$$V_T = \sqrt{V_1^2 + V_2^2} \quad \text{and} \quad \varphi = \tan^{-1}\left(\frac{V_1}{V_2}\right)$$

Resistor in an AC circuit



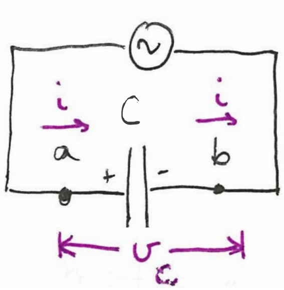
Let $v_R(t) = V_R \cos \omega t$

$$i_R(t) = \frac{v}{R} = \frac{V_R}{R} \cos \omega t = I_0 \cos \omega t$$



- The resistance does not depend on the frequency of the AC source.
- The voltage and current amplitudes are related by Ohm's law: $V_R = I_0 R$
- The phasors for current and voltage are in phase with one another.

Capacitor in an AC circuit



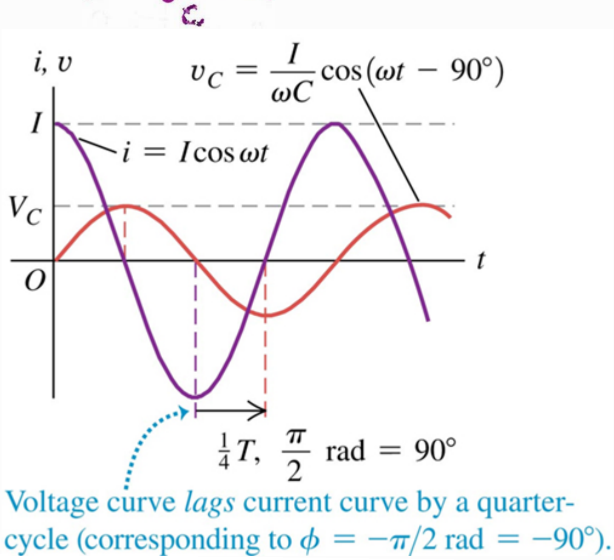
$$v_c(t) = V_c \sin(\omega t)$$

$$\Rightarrow \quad q_c = C v_c = C V_c \sin(\omega t)$$

$$i_c = \frac{dq_c}{dt} = C V_c \omega \cos(\omega t)$$

$$= \frac{V_c}{(1/C\omega)} \cos(\omega t)$$

$$= I_c \sin(\omega t + \phi)$$

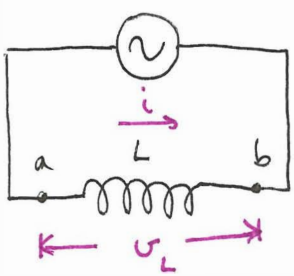


Note that the current through a capacitor is 90° out of phase with voltage.

- When a capacitor is connected to an AC source, the voltage and current amplitudes are related by:

$\boxed{V = I X_c}$ where $\boxed{X_c \equiv \frac{1}{\omega C}}$ is the reactance

Inductor in an AC circuit



$$v_L(t) = V_c \sin(\omega t) = + L \frac{di}{dt}$$

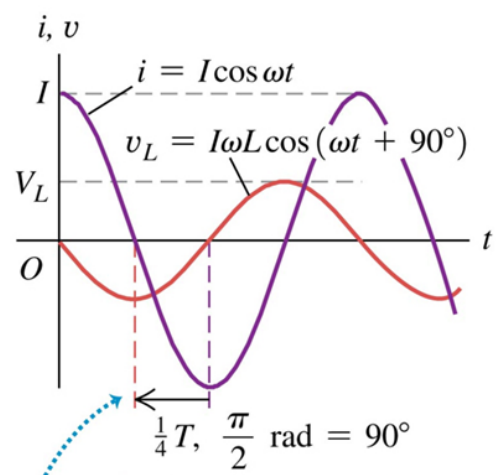
$$\Rightarrow i(t) = \frac{-V_L}{\omega L} \cos(\omega t)$$

Note that the AC current through an inductor is 90° out of phase with voltage.

- When an inductor is connected to an AC source, the voltage and current amplitudes are related by:

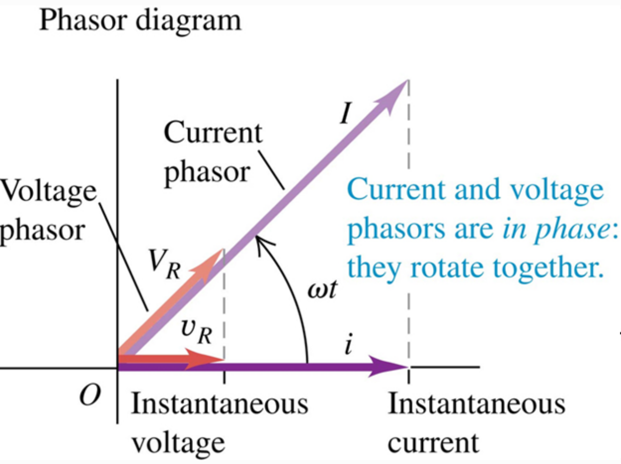
$\boxed{V = I X_L}$ where $\boxed{X_L \equiv \omega L}$ is the reactance

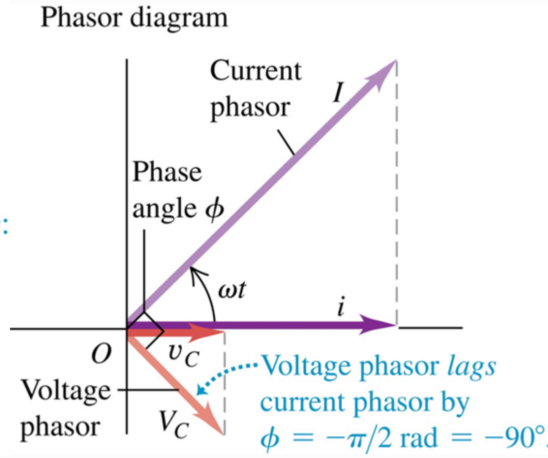
Graphs of current and voltage versus time

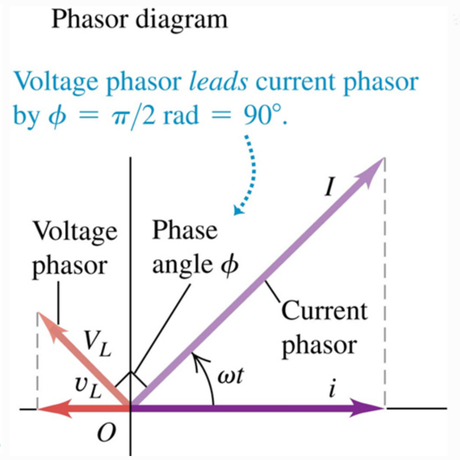


Voltage curve *leads* current curve by a quarter-cycle (corresponding to $\phi = \pi/2$ rad = 90°).

Mnemonic for phase : eLi the iCe man

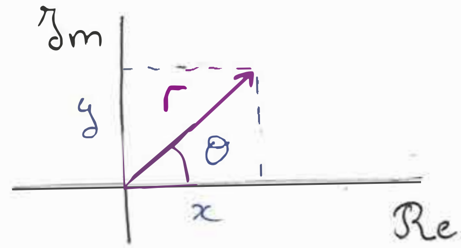






Complex numbers

Phasors can be represented by complex numbers which include both real and imaginary parts:



$$\tilde{z} = x + jy$$

$$= r(\cos\theta + j\sin\theta) = re^{j\theta}$$

$$j \equiv \sqrt{-1} = e^{j\pi/2}$$

with $r = \sqrt{\vec{r}\cdot\vec{r}} = \sqrt{x^2+y^2}$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Multiplying a harmonic function by j increases the phase by $\frac{\pi}{2}$ (rotates the phasor by 90° CCW):

$$\tilde{A}(t) = jae^{j\omega t} = ae^{j\frac{\pi}{2}}e^{j\omega t} = ae^{j(\omega t + \frac{\pi}{2})}$$

Remember:

$$\cos(\omega t) = \sin\left(\omega t + \frac{\pi}{2}\right) \qquad \sin(\omega t) = \cos\left(\omega t - \frac{\pi}{2}\right)$$

Basic complex arithmetic:

If $\tilde{z}_1 = A_1 e^{j\varphi}$ and $\tilde{z}_2 = A_2 e^{j(\varphi+\delta)}$, then:

$$\tilde{z}_1 + \tilde{z}_2 = e^{j\varphi}\left(A_1 + A_2 e^{i\delta}\right)$$

$$= e^{j\varphi}\left((A_1 + A_2\cos\delta) + jA_2\sin\delta\right)$$

$$= e^{j\varphi} M e^{j\beta} = M e^{j(\beta+\varphi)}$$

where $M = \sqrt{(A_1 + A_2\cos\delta)^2 + (A_2\sin\delta)^2}$;

$$\beta = \tan^{-1}\left(\frac{A_2\sin\delta}{A_1 + A_2\cos\delta}\right)$$

If $\tilde{z}_1 = r_1 e^{j\theta_1}$ and $\tilde{z}_2 = r_2 e^{j\theta_2}$, then:

$$\tilde{z}_1\tilde{z}_2 = r_1 e^{j\theta_1} r_2 e^{j\theta_2} = r_1 r_2 e^{j(\theta_2+\theta_1)}$$

$$\frac{\tilde{z}_1}{\tilde{z}_2} = r_1 e^{j\theta_1}\frac{e^{-j\theta_2}}{r_2} = \frac{r_1}{r_2} e^{j(\theta_1-\theta_2)}$$

See math appendix in Y&F or complex numbers in Schaum's.

Complex conjugate:

- whenever a j appears in a complex number, negate it.

$$(a+jb)^* = a - jb \quad \text{and} \quad (Re^{j\theta})^* = Re^{-j\theta}$$

- Multiplying by the complex conjugate always results in a real positive number.

$$(a+jb)(a-jb) = a^2 + b^2$$

- Multiplying by the complex conjugate and taking the square root yields the magnitude of a complex number: $|A| = \sqrt{AA^*}$

Manipulating complex ratios:

It is frequently useful to take a complex ratio, like $\dfrac{a+jb}{c+jd}$ and separate it into real and imaginary components. This allows us to rapidly determine how much of the complex number is "in-phase" and "out-of-phase" with the driving signal.

To do this, multiply top and bottom by the complex conjugate of the bottom.

$$\frac{a+jb}{c+jd}\left(\frac{c-jd}{c-jd}\right) = \frac{(ac+bd)+j(-ad+bc)}{c^2+d^2}$$

$$= \left(\frac{ac+bd}{c^2+d^2}\right) + j\left(\frac{-ad+bc}{c^2+d^2}\right)$$

Complex number worksheet

AC analysis of a resistor (exponential approach)

$$v_R(t) = V_0\cos(\omega t) = \mathrm{Re}\left(V_0 e^{j\omega t}\right)$$

which we just write as $V_0 e^{j\omega t}$

Using $v_R(t) = i(t)R$,

$$i(t) = \frac{v_R(t)}{R} = \frac{V_0}{R}e^{j\omega t} = I_0 e^{j\omega t}$$

$$\boxed{Z_R \equiv \text{impedance} = \frac{v(t)}{i(t)} = R}$$

Impedance, generally $Z = R + jX$, keeps track of both amplitude and phase relationships.

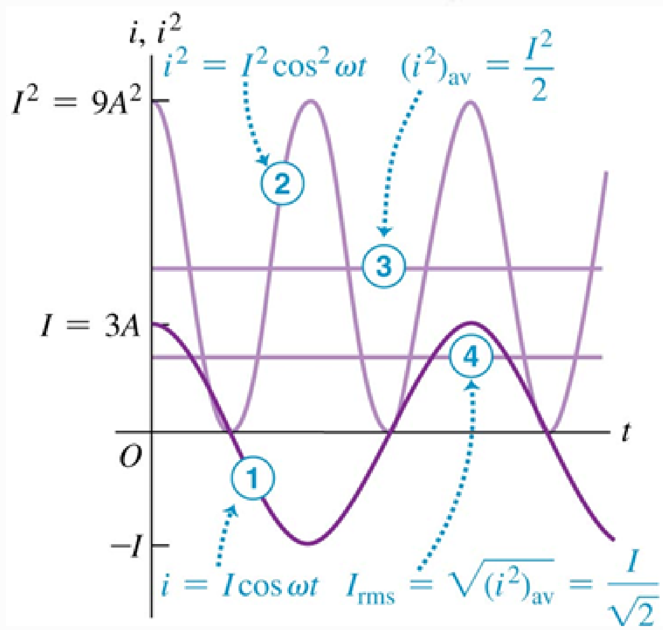
R = resistance is the real (in-phase) part of the $v(t)/i(t)$ ratio.

X = reactance is the out-of-phase part of the ratio (applies to capacitors and inductors).

Root mean square of a sinusoidal voltage:

To calculate the rms of a sinusoidal current:

- square the instantaneous current i
- Take the average (mean) value of i^2
- Take the square root of that average.



$$i(t) = I\cos(\omega t) \Rightarrow I_{rms} = \sqrt{(i^2)_{av}} = \sqrt{\frac{1}{T}\int_0^T i^2(t)\,dt}$$

$$\frac{1}{T}\int_0^T i^2(t)\,dt = \frac{I^2}{T}\int_0^T \cos^2\left(\frac{2\pi}{T}t\right)dt = \frac{I^2}{2}$$

$$\Rightarrow I_{rms} = \frac{I}{\sqrt{2}} \quad ; \quad V_{rms} = \frac{V}{\sqrt{2}}$$

AC analysis of a capacitor (exponential approach)

$$v(t) = V_o \cos(\omega t) = V_o e^{j\omega t}$$

$$v_c(t) = \frac{q_c(t)}{C}$$

$$q(t) = Q_o e^{j\omega t} \Rightarrow V_o e^{j\omega t} = \frac{Q_o e^{j\omega t}}{C}$$

$$\Rightarrow Q_o = V_o C$$

$$i(t) = \frac{dq}{dt} = j\omega V_o C e^{i\omega t}$$

$$Z_c = \frac{v(t)}{i(t)} = \frac{V_o e^{j\omega t}}{j\omega V_o C e^{j\omega t}} = \frac{1}{j\omega C}$$

phase change

$$\boxed{X_c = \frac{1}{\omega C}}$$

$v(t) = i(t) Z$: treat Z like R in circuit.

AC analysis of an inductor (exponential approach)

$$v(t) = V_o \cos(\omega t) = \mathrm{Re}(V_o e^{j\omega t})$$

$$v_L(t) = -L \frac{di(t)}{dt}$$

$$\Rightarrow i(t) = -\frac{1}{L}\int v_L(t)\,dt = -\frac{V_o}{L}\int e^{j\omega t}\,dt = \frac{-1}{j\omega L} V_o e^{j\omega t}$$

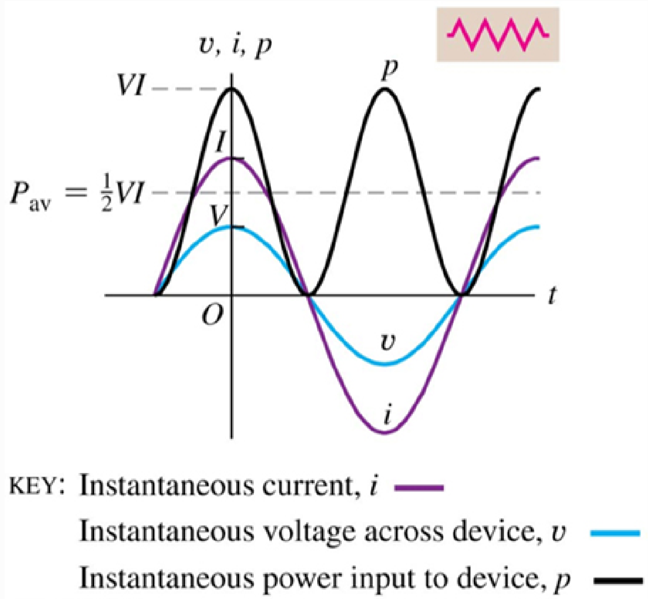
$$Z_L = \frac{v(t)}{i(t)} = \frac{V_o e^{j\omega t}}{\frac{-1}{j\omega L} V_o e^{j\omega t}} = -j\omega L$$

$$\boxed{X_L = \omega L}$$

Power in AC circuits

The instantaneous power flowing into or out of a device is the product of the voltage and current.

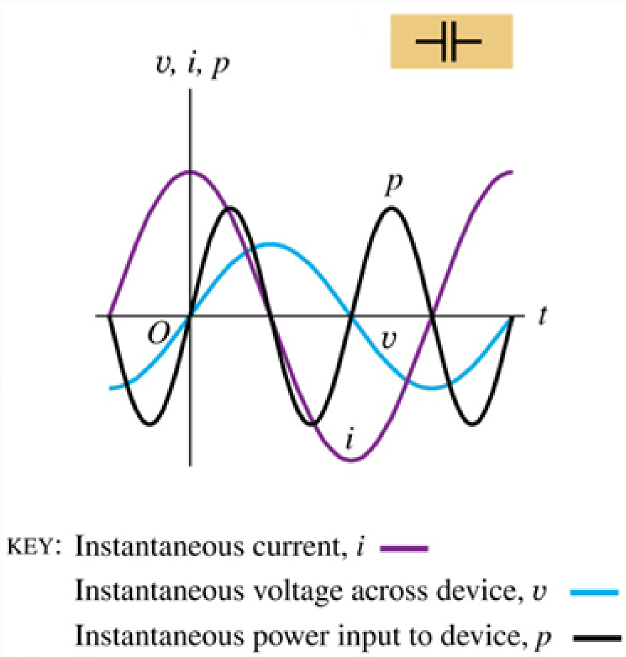
$$\boxed{p(t) = i(t)\,v(t)}$$



- For a resistor:

$$p_R(t) = iv = \frac{V_R^2}{R}\sin^2(\omega t)$$

$$P_{R\,avg} = \frac{1}{T}\int_0^T \frac{V_R^2}{R}\sin^2(\omega t)\,dt = \frac{1}{2}\frac{V_R^2}{R} = \frac{V_{rms}^2}{R}$$



- In capacitors:

The current is 90° out-of-phase with the voltage.

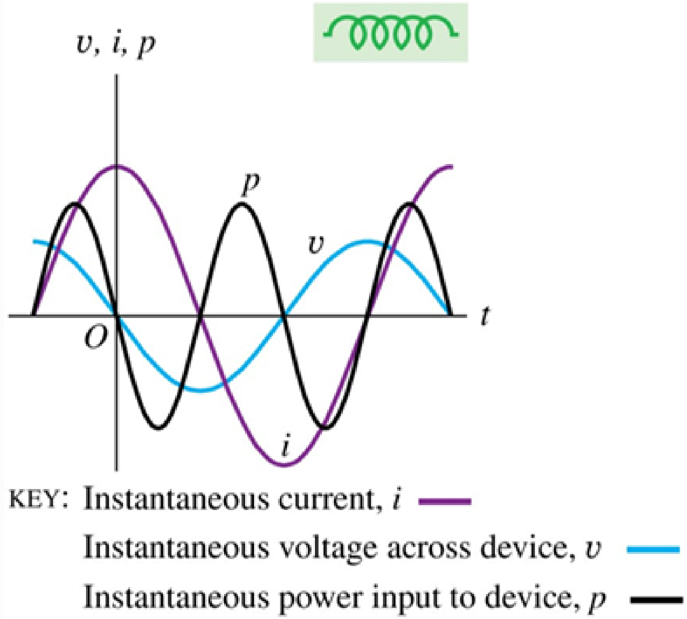
$$p_C(t) = i_C v_C = V_C^2\,\omega C\sin(\omega t)\cos(\omega t)$$

$$P_{C\,avg} = 0$$

As charge oscillates, energy flows into, and then out, of a capacitor.

• In inductors:

The current and voltage are also 90° out of phase with each other.



$$p_L(t) = i_L v_L = \frac{V_L^2}{X_L} \sin(\omega t)\cos(\omega t)$$

$$P_{L_{avg}} = 0$$

As current oscillates, energy flows into, and then out, of an inductor.

Summary

- Phasors allow us to visualize oscillating systems.
- Exponential notation allows us to do math for oscillating systems more easily.
- Neither an ideal capacitor nor inductor dissipate energy in circuits! ($E_{IN} = E_{OUT}$)
- $v(t) = i(t) Z$; $Z_R = R$, $Z_C = \frac{1}{j\omega C}$, $Z_L = j\omega L$
- Multiplying by j advances the phase by $\pi/2$
- Multiplying by $-j = \frac{1}{j}$ delays the phase by $\pi/2$
- $V_0 = I_0 |Z|$; $|Z| = \sqrt{Z Z^*}$